Upper bounds on violation of Bell-type inequalities by a multipartite quantum state

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Abstract

We present the new exact upper bounds on the maximal Bell violation for the generalized N-qubit GHZ state, the N-qudit GHZ state and, in general, for an arbitrary N-partite quantum state, possibly infinite-dimensional. Our results indicate that, for an N-partite quantum state of any Hilbert space dimension, violation of any Bell-type inequality (either on correlation functions or on joint probabilities) with S settings and any number of outcomes at each site cannot exceed $(2S-1)^{N-1}$.

1 Introduction

Multipartite Bell-type inequalities[1] are now widely used in many schemes of quantum information processing. However, the exact upper bounds on quantum Bell violations are well known only in case of bipartite correlation Bell-type inequalities where, independently on a Hilbert space dimension of a bipartite quantum state and numbers of measurement settings per site, quantum violations cannot exceed [2, 3] the Grothendieck constant.

Bounds on violation by a bipartite quantum state of Bell-type inequalities for joint probabilities have been recently intensively discussed in the literature both computationally [4] and theoretically, see [5, 6, 7] and references therein.

For an arbitrary N-partite quantum state, the exact upper bounds on the maximal quantum Bell violation have not been reported in the literature but it has been argued in [5] that, via increasing of a Hilbert space dimension of some tripartite quantum states, these states "can lead to arbitrarily large violations of Bell inequalities"[2].

In this concise presentation on our results in [8-10], we present the exact upper bounds on violation by N-partite quantum states of any Bell-type inequality, either on correlation functions or on joint probabilities. Specified for $N = 2, 3$, our new general results improve the bipartite upper bounds reported in [6, 7] and also clarify the range of applicability of the tripartite lower estimate found in [5].

[1]On the general framework for multipartite Bell-type inequalities, see [1].
[2]Cited according to [5]

2 Some new Hilbert space notions

In this section, we shortly introduce some new tensor Hilbert space notions [8-10] needed for our further consideration.

Source operators. For a state ρ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, denote by $T_{S_1\times\cdots\times S_N}^{(\rho)}$ any of its self-adjoint trace class dilations to space $\mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N}$.

We refer to dilation $T_{S_1\times\cdots\times S_N}^{(\rho)}$ as *an* $S_1 \times \cdots \times S_N$*-setting source operator* for state ρ and set $T_{1\times\cdots\times 1}^{(\rho)} := \rho$. For any source operator T, it trace $\mathrm{tr}[T] = 1$.

Proposition 1 *For any* N*-partite quantum state* ρ*, possibly infinite-dimensional, and any positive integers* $S_1, ..., S_N \geq 1$*, an* $S_1 \times \cdots \times S_N$*-setting source operator* $T_{S_1\times\cdots\times S_N}^{(\rho)}$ *exists.*

Tensor positivity. We refer to a trace class operator W on a Hilbert space space $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, $m \geq 1$ as *tensor positive* and denote this by $W \overset{\otimes}{\geq} 0$ if

$$\mathrm{tr}[W\{X_1 \otimes \cdots \otimes X_m\}] \geq 0, \tag{1}$$

for any positive bounded linear operators $X_1, ..., X_m$ on spaces $\mathcal{G}_1, .., \mathcal{G}_m$, respectively.

The covering norm. For a self-adjoint trace class operator W on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, we call a tensor positive trace class operator W_{cov} on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$ satisfying relations

$$W_{cov} \pm W \overset{\otimes}{\geq} 0, \tag{2}$$

as *a trace class covering of W.*

Proposition 2 *For any operator*[3] $W \in \mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}^{(sa)}$*, its trace class covering* W_{cov} *exists and relation*

$$\|W\|_{cov} := \inf_{W_{cov}\in\mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}} \mathrm{tr}[W_{cov}] \tag{3}$$

defines on space $\mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}^{(sa)}$ *a norm, the covering norm, with properties:*

$$\begin{aligned} |\mathrm{tr}[W]| \quad &\leq \quad \|W\|_{cov} \leq \|W\|_1\,, \\ W \overset{\otimes}{\geq} 0 \quad &\Rightarrow \quad \|W\|_{cov} = \mathrm{tr}[W]. \end{aligned} \tag{4}$$

3 LqHV simulation of a quantum correlation scenario

For a state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, consider an N-partite correlation scenario[4] $\mathcal{E}_\rho$ where each n-th of N parties performs S_n measurements with outcomes[5] $\lambda_n \in \Lambda_n := \{\lambda_n^{(1)}, ..., \lambda_n^{(L_n)}\}$.

[3]Here, $\mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ and $\mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}^{(sa)}$ denote, correspondingly, the space of all trace class operators and the space of all self-adjoint trace class operators on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$.

[4]On the general framework for the probabilistic description of multipartite correlation scenarios, see [8].

[5]For simplicity, we consider here only discrete outcomes. This does not, however, imply any restriction on our main results since, as it has been proved in [10], the latter hold for outcomes of any spectral type, discrete or continuous.

We label each measurement at n-th site by a positive integer $s_n = 1, ..., S_n$, and each of N-partite joint measurements, induced by this correlation scenario - by an N-tuple $(s_1, ..., s_N)$ where n-th component refers to a marginal measurement at n-th site.

Let, under the correlation scenario $\mathcal{E}_\rho$, each quantum measurement s_n at n-th site be represented on $\mathcal{H}_n$ by a POV measure $\mathrm{M}_n^{(s_n)}$. For a joint measurement $(s_1, ..., s_N)$ under scenario $\mathcal{E}_\rho$, expression

$$\begin{aligned} & P_{(s_1,...,s_N)}^{(\mathcal{E}_\rho)}(\lambda_1, ..., \lambda_N) \\ = \; & \mathrm{tr}[\rho\{\mathrm{M}_1^{(s_1)}(\lambda_1) \otimes \cdots \otimes \mathrm{M}_N^{(s_N)}(\lambda_N)\}] \end{aligned} \tag{5}$$

specifies the joint probability $P_{(s_1,...,s_N)}^{(\mathcal{E}_\rho)}(\lambda_1, ..., \lambda_N)$ that each n-th party observes an outcome $\lambda_n \in \Lambda_n$.

If $T_{S_1 \times \cdots \times S_N}^{(\rho)}$ is an $S_1 \times \cdots \times S_N$- setting source operator[6] for state ρ, then, due to property $\mathrm{M}_n^{(s_n)}(\Lambda_n) = \mathbb{I}_{\mathcal{H}_n}$, each probability (5) constitutes the corresponding marginal of the normalized real-valued distribution

$$\begin{aligned} & \mathrm{tr}[T_{S_1 \times \cdots \times S_N}^{(\rho)}\{\mathrm{M}_1^{(1)}(\lambda_1^{(1)}) \otimes \cdots \otimes \mathrm{M}_1^{(S_1)}(\lambda_1^{(S_1)}) \otimes \\ & \cdots \otimes \mathrm{M}_N^{(1)}(\lambda_N^{(1)}) \otimes \cdots \otimes \mathrm{M}_N^{(S_N)}(\lambda_N^{(S_N)})\}], \\ \lambda_n^{(s_n)} \in \; & \Lambda_n, \quad s_n = 1, ..., S_n, \; n = 1, ..., N. \end{aligned} \tag{6}$$

This implies.

Theorem 1 *[10] For every N-partite quantum state ρ and any positive integers $S_1, ..., S_N \geq 1$, each $S_1 \times \cdots \times S_N$ - setting correlation scenario $\mathcal{E}_\rho$ admits a local quasi hidden variable (LqHV) model*

$$\begin{aligned} P_{(s_1,...,s_N)}^{(\mathcal{E}_\rho)}(\lambda_1, ..., \lambda_N) &= \int_\Omega P_1^{(s_1)}(\lambda_1 \mid \omega) \cdot \ldots \cdot P_N^{(s_N)}(\lambda_N \mid \omega) \; \nu_{\mathcal{E}_\rho}(\mathrm{d}\omega), \\ s_1 &= 1, ..., S_1, ..., s_N = 1, ..., S_N, \end{aligned} \tag{7}$$

where $\nu_{\mathcal{E}_\rho}$ is a normalized bounded real-valued[7] measure of some variables $\omega \in \Omega$ and $P_n^{(s_n)}(\cdot \mid \omega)$, $\forall s_n, \forall n$, are conditional probabilities.

Thus, an arbitrary N-partite state ρ does not need to admit an $S_1 \times \cdots \times S_N$-setting LHV description [8] but it necessarily admits an $S_1 \times \cdots \times S_N$-setting LqHV description.

4 Bell-type inequalities

For a general $S_1 \times ... \times S_N$-setting correlation scenario $\mathcal{E}$, consider a linear combination

$$\sum_{s_1,...,s_N} \left\langle \; \psi_{(s_1,...,s_N)}(\lambda_1, ..., \lambda_N) \right\rangle_{\mathcal{E}} \tag{8}$$

[6] See in section 2.

[7] Recall that, in an LHV model, measure $\nu_{\mathcal{E}_\rho}$ must be positive.

of averages

$$\Big\langle \psi_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N)\Big\rangle_{\mathcal{E}} \quad (9)$$
$$:= \sum_{\lambda_1\in\Lambda_1,...,\lambda_N\in\Lambda_N} \psi_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N)\ P^{(\mathcal{E})}_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N),$$

specified by a family $\{\psi_{(s_1,...,s_N)}\}$ of bounded real-valued functions on set $\Lambda := \Lambda_1 \times \cdots \times \Lambda_N$. For a particular choice of functions $\{\psi_{(s_1,...,s_N)}\}$, averages in (9) may reduce either to joint probabilities or to correlation functions.

In an *LHV* case, any linear combination (8) of averages satisfies the following tight[8] LHV constraints [1]:

$$\mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}} \leq \sum_{s_1,...,s_N} \Big\langle \psi_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N) \Big\rangle_{\mathcal{E}_{lhv}} \leq \mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}}, \quad (10)$$

with the LHV constants

$$\mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}} = \sup_{\lambda_n^{(s_n)}\in\Lambda_n,\ \forall s_n,\forall n} \sum_{s_1,...,s_N} \psi_{(s_1,...,s_N)}(\lambda_1^{(s_1)},...,\lambda_N^{(s_N)}), \quad (11)$$
$$\mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}} = \inf_{\lambda_n^{(s_n)}\in\Lambda_n,\ \forall s_n,\ \forall n} \sum_{s_1,...,s_N} \psi_{(s_1,...,s_N)}(\lambda_1^{(s_1)},...,\lambda_N^{(s_N)}).$$

The general LHV constraint form (10) incorporates as particular cases both - the LHV constraints on correlation functions and the LHV constraints on joint probabilities.

A Bell-type inequality is any of the tight linear LHV constraints (10) that may be violated in a non-LHV case.

5 Quantum violations

For an arbitrary $S_1 \times \cdots \times S_N$-setting quantum scenario $\mathcal{E}_\rho$ specified by joint probabilities (5), every linear combination (8) of its averages satisfies the following analogs [10] of the LHV constraints (10):

$$\mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}} - \frac{\Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N}-1}{2}(\mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}} - \mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}}) \quad (12)$$
$$\leq \sum_{s_1,...,s_N} \Big\langle \psi_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N) \Big\rangle_{\mathcal{E}_\rho}$$
$$\leq \mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}} + \frac{\Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N}-1}{2}(\mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}} - \mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}}),$$

where

$$\Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N} = \sup_{\{\psi_{(s_1,...,s_N)}\}} \frac{1}{\mathcal{B}_{\{\psi_{(s_1,...,s_N)}\}}} \Big| \sum_{s_1,...,s_N} \Big\langle \psi_{(s_1,...,s_N)}(\lambda_1,...,\lambda_N) \Big\rangle_{\mathcal{E}_\rho} \Big|, \quad (13)$$

[8] A tight LHV constraint is not necessarily extreme, see [1] for details.

is *the maximal violation* by state ρ of any Bell-type inequality (either on correlation functions or on joint probabilities) specified for settings up to setting $S_1 \times \cdots \times S_N$ and outcomes in set $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N$. In (13),

$$\mathcal{B}_{\{\psi_{(s_1,...,s_N)}\}} := \max\{|\mathcal{B}^{\sup}_{\{\psi_{(s_1,...,s_N)}\}}|, |\mathcal{B}^{\inf}_{\{\psi_{(s_1,...,s_N)}\}}|\}. \tag{14}$$

For short, we further refer to parameter $\Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N}$ as the maximal $S_1 \times \cdots \times S_N$- setting Bell violation for state ρ and outcomes in Λ.

Using the new Hilbert space notions specified in section 2, we have the following general statements.

Theorem 2 *[10] For an arbitrary N-partite quantum state ρ, possibly infinite-dimensional, and any positive integers $S_1, ..., S_N \geq 1$, the maximal $S_1 \times \cdots \times S_N$- setting Bell violation $\Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N}$ satisfies relation*

$$1 \leq \Upsilon^{(\rho,\Lambda)}_{S_1\times\cdots\times S_N} \leq \inf_{T^{(\rho)}_{S_1\times\cdots\times \underset{\uparrow n}{1}\times\cdots\times S_N},\ \forall n} ||T^{(\rho)}_{S_1\times\cdots\times \underset{\uparrow n}{1}\times\cdots\times S_N}||_{cov}, \tag{15}$$

for any outcome set $\Lambda = \Lambda_1 \times \cdots \times \Lambda_N$. Here, $\|\cdot\|_{cov}$ is the covering norm and infimum is taken over all source operators $T^{(\rho)}_{S_1\times\cdots\times \underset{\uparrow n}{1}\times\cdots\times S_N}$ for all $n = 1, ..., N$.

Corollary 1 *[10] If a state ρ has a tensor positive source operator $T^{(\rho)}_{S_1\times\cdots\times \underset{\uparrow n}{1}\times\cdots\times S_N}$ then it admits an $S_1 \times \cdots \times S_N$- setting LHV description for any finite number S_n of measurement settings at site "n".*

Corollary 2 *[10] If a state ρ has a tensor positive source operator $T^{(\rho)}_{S_1\times\cdots\times S_N}$, then this state admits an $S_1 \times \cdots \times \widetilde{S}_n \times \cdots \times S_N$- setting LHV description for any finite number $\widetilde{S}_n$ of settings at each n-th site.*

6 Numerical bounds

The general analytical upper bound (15) allows us to find [10] the following new exact numerical bounds on the maximal quantum Bell violations.

- For the two-qubit singlet ρ_{singlet}, the maximal Bell violation

$$\Upsilon^{(\rho_{\text{singlet}},\Lambda)}_{S\times 2} \leq \sqrt{3}, \quad S \geq 2, \tag{16}$$

  for any outcome set $\Lambda = \Lambda_1 \times \Lambda_2$, in particular, for any number of outcomes at each site. Note that, due to the seminal results of Tsirelson[9] and Fine [10], the maximal Bell violation $\Upsilon^{(\rho,\Lambda)}_{2\times 2} \leq \sqrt{2}$, for any bipartite state ρ and any outcome set $\Lambda = \{\lambda_1^{(1)}, \lambda_1^{(2)}\} \times \{\lambda_2^{(1)}, \lambda_2^{(2)}\}$ (dichotomic measurements). The maximal violation by the singlet of any *correlation* Bell-type inequality is given [3] by the Grothendieck constant $\sqrt{2} \leq K_G(3) \leq 1.5163...$.

[9] Tsirelson B.: *J. Soviet Math.* **36**, 557 (1987).
[10] Fine A.: *Phys. Rev. Lett.* **48**, 291 (1982)

- For the N-qudit GHZ state

$$\frac{1}{\sqrt{d}}\sum_{j=1}^{d}\underbrace{|j\rangle\otimes\cdots\otimes|j\rangle}_{N}, \tag{17}$$

violation of any Bell-type inequality for S settings and any number of outcomes per site cannot exceed

$$\begin{aligned} & \min\{(2S-1)^{N-1},\ 1+2^{N-1}(d-1)\} \\ \leq\ & 1+2^{N-1}\left[\min\{S^{N-1},d\}-1\right]. \end{aligned} \tag{18}$$

- For the generalized N-qubit GHZ state

$$\sin\varphi\ |1\rangle^{\otimes N}+\cos\varphi\ |2\rangle^{\otimes N}, \tag{19}$$

violation of any Bell-type inequality for S settings and any number of outcomes per site is upper bounded by

$$1+2^{N-1}\left|\sin 2\varphi\right|. \tag{20}$$

- For an arbitrary state ρ on $\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_N}$, the maximal Bell violation in case of S_n settings and any number of outcomes at each n-th site is upper bounded by

$$1+2^{N-1}\left[\min\left\{\frac{S_1\cdot\ldots\cdot S_N}{\max_n S_n},\frac{d_1\cdot\ldots\cdot d_N}{\max_n d_n}\right\}-1\right]. \tag{21}$$

If $S_1=\ldots=S_N=S$, then the maximal Bell violation cannot exceed

$$\begin{aligned} & \min\{(2S-1)^{N-1},\ 1+2^{N-1}(\frac{d_1\cdot\ldots\cdot d_N}{\max_n d_n}-1)\} \\ \leq\ & 1+2^{N-1}\left[\min\{S^{N-1},\frac{d_1\cdot\ldots\cdot d_N}{\max_n d_n}\}-1\right]. \end{aligned} \tag{22}$$

From this N-partite bound it follows that violation by an arbitrary N-partite quantum state, possibly infinite-dimensional, of any Bell inequality for S measurement settings and any number of outcomes per site cannot exceed $(2S-1)^{N-1}$.

6.1 Bipartite and tripartite bounds

For $N=2$, the general upper bound (21) implies the following *bipartite upper bound* [10] on the maximal Bell violation

$$\Upsilon_{S_1\times S_2}^{(\rho,\Lambda)}\leq 2\min\{S_1,S_2,d_1,d_2\}-1 \tag{23}$$

for any quantum state ρ on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$ and any outcome set $\Lambda=\Lambda_1\times\Lambda_2$. This new bipartite upper bound improves:

- for (i) $d_1=d_2=2$, $L_1=L_2=2$, and (ii) $d_1=d_2\leq L_1L_2\ (K_G+1)$, $\forall L_1,L_2$, the corresponding numerical upper bounds on the maximal Bell violation (in our notation):

$$\begin{aligned} \text{(i)}\quad \Upsilon_{S_1\times S_2}^{(\rho,\Lambda)} &\leq 2K_G+1, \qquad L_1=L_2=2, \\ \text{(ii)}\quad \Upsilon_{S_1\times S_2}^{(\rho,\Lambda)} &\leq 2L_1L_2(K_G+1)-1, \qquad \forall L_1,L_2, \end{aligned} \tag{24}$$

found in [6] for any bipartite quantum state ρ and L_1,L_2 outcomes at Alice's and Bob's sites. Here, $K_G=\lim_{n\to\infty}K_G(n)\in[1.676...,1.782...]$ is the Grothendieck constant;

- the approximate bipartite estimate

$$\Upsilon_{S\times S}^{(\rho,\Lambda)} \preceq \min\{S,d\}, \qquad \forall\Lambda, \tag{25}$$

derived in [7] up to an unknown universal constant for any bipartite state ρ on $\mathbb{C}^d\otimes\mathbb{C}^d$;

For $N=3$, the general upper bound (22) implies the following *tripartite upper bound* [10] on the maximal Bell violation:

$$\Upsilon_{S\times S\times S}^{(\rho,\Lambda)} \leq \min\{(2S-1)^2,\ 4\frac{d_1d_2d_3}{\max_n d_n}-3\}, \tag{26}$$

for any tripartite state ρ on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ and any outcome set $\Lambda=\Lambda_1\times\Lambda_2\times\Lambda_3$.

For a state ρ on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d$, bound (26) implies

$$\begin{aligned}\Upsilon_{S\times S\times S}^{(\rho,\Lambda)} &\leq \min\{(2S-1)^2,\ 4d^2-3\} \\ &\leq 4(\min\{S,d\})^2-3.\end{aligned} \tag{27}$$

From (26) it follows – the approximate lower estimate $\succeq\sqrt{d}$ found in [5] for violation of some correlation Bell-type inequality by some tripartite state on $\mathbb{C}^d\otimes\mathbb{C}^D\otimes\mathbb{C}^D$ is meaningful if only in this correlation Bell-type inequality a number of settings per site satisfies relation

$$(2S-1)^2\succeq\sqrt{d}. \tag{28}$$

7 Conclusions

Via some new Hilbert space notions and a new simulation approach, *the LqHV approach,* to the description of any quantum correlation scenario, we have derived the analytical upper bound (15) on the maximal Bell violation by an N-partite quantum state. This has allowed us:

- to single out N-partite quantum states admitting an $S_1\times\cdots\times S_N$-setting LHV description;

- to find the new numerical upper bounds on Bell violations for some concrete N-partite states generally used in quantum information processing;

- to prove that violation by an arbitrary N-partite quantum state, possibly infinite-dimensional, of any Bell inequality (either on correlation functions or on joint probabilities) for S measurement settings and any number of outcomes per site cannot exceed $(2S-1)^{N-1}$;

- to improve the bipartite upper bounds reported in [6, 7];

- to show that, for an "arbitrarily large" tripartite quantum Bell violation argued in [5] to be reached, not only a Hilbert space dimension d but also a number S of settings per site in the corresponding tripartite Bell-type inequality must be large and the required growth of S with respect to d is given by (28).

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